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# Refinable Functions with Compact Support

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In this paper a refinable and blockwise polynomial with compact support is shown to be a finite linear combination of a box-spline and its translates (Theorems 1 and 2). Zak transform is used to give an upper bound for the regularity degree of a refinable function with compact support (Theorem 3). 

## 1. INTRODUCTION AND RESULTS

For an integer $m \geqslant 2$, a compactly supported function $f$ is called *m-refinable* if there exists a sequence $\{c_j\}$ of finite length such that

$$f(x) = \sum_j c_j f(mx - j). \tag{1}$$

A function is called refinable if $f$ is $m$-refinable for some integer $m \geqslant 2$. Refinable function arises in dyadic interpolation, in the construction of non-differentiable function, and mainly in multiresolution. It has a strong impact on the theory and application of wavelets [D1]. In 1992, Daubechies and Lagarias [DL] proved the nonexistence of $C^\infty$ refinable function with compact support in one dimension, and Cavaretta *et al.* [CDM] extended their result to higher dimensions by using the matrix method. Recently Lawton *et al.* [LLS] further proved that a refinable spline is a finite combination of $B$-splines in one dimension. The purpose of this paper is to extend their result to higher dimensions and to give an upper bound for the regularity degree of a refinable function by using the Zak transform.

To these aims, we introduce some definitions. A function $f$ is called a *blockwise polynomial* if there exists a simplex decomposition $\{\Delta_j\}_{j=1}^N$ to supp $f$, the *supporting set* of $f$, such that $f$ is a polynomial on every simplex $\Delta_j$, $1 \leqslant j \leqslant N$. Hereafter $\Delta^0 = \{(x_1, x_2, ..., x_n) \in R^n;\ 0 \leqslant x_j \leqslant 1,\ \sum_{j=1}^n x_j \leqslant 1\}$,

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is called *standard simplex* on $R^n$, and a simplex $\Delta$ is a nonsingular affine transform of standard simplex, i.e., $\Delta = A\Delta^0 + c$, for some nonsingular matrix $A$ and $c \in R^n$. We say that $\{\Delta_j\}_{j=1}^N$ is a simplex decomposition of a bounded set $E$ if $\bigcup_{j=1}^N \Delta_j \supset E$, $\Delta_j$ is simplex for every $j$, and $\Delta_j \cap \Delta_{j'}$ has Lebesgue measure zero when $j \neq j'$.

Let $\Xi = (a_1, a_2, ..., a_s)$ be an $s \times n$ matrix with integral entries and of full rank $n$. Define the *box-spline* $B_\Xi$ with the help of Fourier transform by

$$\hat{B}_\Xi(\xi) = \prod_{j=1}^{s} \frac{e^{ia_j\xi} - 1}{ia_j\xi}. \tag{2}$$

When $\Xi = (1, 1, ..., 1)$ in one dimension the box spline $B_\Xi$ defined above is called the *B-spline*. Hereafter, *Fourier transform* $\hat{f}$ of an integrable function $f$ is defined by $\hat{f}(\xi) = \int_{R^n} e^{-ix\xi} f(x)\, dx$. A Laurent polynomial $R(z)$ is said to be *m closed* if $R(z^m)/R(z)$ is a Laurent polynomial.

In this paper we will prove the following theorem, which extends Lawton *et al.*'s result to higher dimensions.

THEOREM 1. *Let $n \geqslant 2$. Let $f$ be a compactly supported blockwise polynomial. Then $f$ is m-refinable if and only if*

$$f(x) = P(D)\left(\sum_k d_k B_\Xi\left(x - k - \frac{l}{m-1}\right)\right),$$

*where $P(D)$ is a homogeneous differential operator, $B_\Xi$ is a box-spline defined by* (2), $(\sum_k d_k z^k) \prod_{i=1}^s (z^{a_j} - 1)$ *is m-closed, and $l$ is an integer.*

In one dimension we will prove Lawton *et al.*'s result under weaker conditions. A compactly supported function on $R$ is *piecewise smooth* if there exist an integer $N$ and $a_1 < a_2 < \cdots < a_{N+1}$ such that $f$ is smooth on every subinterval $(a_j, a_{j+1})$, $1 \leqslant j \leqslant N$, and $\operatorname{supp} f \subset [a_1, a_{N+1}]$.

THEOREM 2. *Let $n = 1$ and let $f$ be a piecewise smooth function with compact support. Then $f$ is m-refinable if and only if*

$$f(x) = \sum_k d_k B\left(x - k - \frac{l}{m-1}\right)$$

*where $l$ is a fixed integer and $k \in Z$, $B(x)$ is a B-spline, and $(z-1)^s \sum_k d_k z^k$ is m-closed.*

The Zak transform is a very important tool to study Gabor transform [D2]. After establishing a formula of the Zak transform of refinable function, we estimate an upper bound for the regularity degree of refinable function.

THEOREM 3. *Let $f$ be a nonzero compactly supported function which satsifies* (1). *Denote the set of homogeneous differential operators $P(D)$ such that $P(D)f$ is continuous by $\mathscr{P}$. Then the dimension of $\mathscr{P}$ does not exceed $\#\{j, c_j \neq 0\}$, where $\# E$ denotes the cardinality of the set $E$.*

Compared with the estimate of regular degree in [DL] and [CDM], this theorem has two improvements. One is that we can consider different regularities in different directions to $f$ instead of $f \in C^k$ for some $k$. The other one is that the regularity degree is estimated by the cardinality of all nonzero $c_j$ instead of by the length of $\{c_j\}$. It obviously implies the nonexistence of the $C^\infty$ refinable function $f$ with $\{c_j\}$ in (1) having finite length, and reproves the results of Daubechies and Lagarias [DL] and Cavaretta *et al.*'s result [CDM].

Observe that the dimension of $\mathscr{P}$ in Theorem 3 is $\binom{n+s}{n}$ when $f$ belongs to $C^s(R^n)$. Therefore we get

COROLLARY 1. *Let a compactly supported function $f$ satsify* (1). *If $f \in C^s(R^n)$, then*

$$\binom{n+s}{n} \leqslant \#\{j, c_j \neq 0\}.$$

The paper is organized as follows. The proofs of Theorems 1 and 2 are given in Section 2, and the proof of Theorem 3 is given in Section 3.

## 2. PROOFS OF THEOREMS 1 AND 2

To prove Theorems 1 and 2, we need some preliminaries. A polynomial $P$ is called a *principal homogeneous polynomial* if there exist a natural number $K$ and $A_j \in R^n$ $(1 \leqslant j \leqslant K)$ such that $P(\xi) = \prod_{j=1}^K A_j \xi$. $T(\xi) = \sum_j a_j e^{i b_j \xi}$ for real $b_j$ and complex $a_j$ is called a *generalized trigonometric polynomial.*

LEMMA 1. *Let $f$ be a blockwise polynomial with compact support. Then*

$$\hat{f}(\xi) = \sum_j \frac{T_j(\xi)}{P_j(\xi)}, \tag{3}$$

*where each $T_j$ is a generalized trigonometric polynomial and each $P_j$ is a principal homogeneous polynomial.*

*Proof.* Obviously it suffices that (3) holds for a polynomial $f$ on the standard simplex $\Delta^0$. Integrating by parts, we get

$$\int_{\Delta^0} e^{-ix\xi} f(x)\, dx = -\frac{1}{i\xi_n} \int_{\Delta^0} e^{-ix\xi} \frac{\partial}{\partial x_n} f(x)\, dx$$
$$+ \frac{e^{-i\xi_n}}{i\xi_n} \int_{\Delta^{0\prime}} e^{-ix'(\xi' - \xi_n e)} f(x', 1 - \|x'\|)\, dx'$$
$$- \frac{1}{i\xi_n} \int_{\Delta^{0\prime}} e^{-ix'\xi'} f(x', 0)\, d\xi',$$

where $\Delta^{0\prime} = \{x' : x_j \geqslant 0, \sum_{j=1}^{n-1} x_j \leqslant 1\}$, $x' = (x_1, ..., x_{n-1})$ for $x = (x_1, ..., x_n)$, $e = (1, ..., 1)$, and $\|x'\| = \sum_{j=1}^{n-1} x_j$. Lemma 1 follows by a finite number of iterations of the above procedure. ∎

LEMMA 2. *Suppose $\{x_j\}$ are finitely distinct real numbers. If $\sum_j c_j e^{ix_j r} \to 0$ as $r \to +\infty$, then $c_j = 0$.*

*Proof.* We prove the lemma by induction on the cardinality of $N = \#\{x_j\}$. Obviously the conclusion holds when $N = 1$ since $|e^{-ix_j r}| = 1$ for all $r$. Inductively we assume that the conclusion holds for all $N \leqslant k$. Let $g(r) = \sum_{j=1}^{k+1} c_j e^{i(x_j - x_1) r}$. Observe that for every $s > 0$,

$$\frac{1}{s} \int_r^{r+s} g(t)\, dt - g(r) = -\sum_{j=2}^{k+1} c_j e^{i(x_j - x_1) r} \left\{1 - \frac{e^{i(x_j - x_1) s} - 1}{is(x_j - x_1)}\right\} \to 0$$

as $r \to +\infty$. Hence $c_j = 0$ for all $2 \leqslant j \leqslant k+1$ by inductive hypothesis and $s$ is arbitrary and $c_1 = 0$ also. ∎

LEMMA 3. *Let $P_j$ $(j = 1, 2)$ be two nonzero homogeneous polynomials and let $T_j$ $(j = 1, 2)$ be two nonzero trigonometric polynomials. If*

$$P_1(\xi)\, T_1(\xi) = e^{i\alpha\xi} P_2(\xi)\, T_2(\xi) \tag{4}$$

*holds for some $\alpha \in R^n$, then $\alpha \in Z^n$, $P_1(\xi) = CP_2(\xi)$, and $T_1(\xi) = C^{-1} e^{i\alpha\xi} T_2(\xi)$ for some complex number $C$.*

*Proof.* Define the *difference operator* $\delta_j$ with step $2\pi e^j$ by $\delta_j f(\xi) = f(\xi) - f(\xi + 2\pi e^j)$ where $e^j \in R^n$ is the vector with the $j$th component 1 and all other components 0. Observe that $\delta_j T_1 = \delta_j T_2 = 0$, $\deg(\delta_j P_1) \leqslant \deg P_1 - 1$, and $\deg(\delta_j P_2) \leqslant \deg P_2 - 1$. On the other hand, $\deg \delta_j P_1 = \deg P_1 - 1$ for at least one $j$. Therefore we can find difference operators $\delta_{j(s)}$ $(1 \leqslant s \leqslant \deg P_1)$ such that $\delta_{j(\deg P_1)} \cdots \delta_{j(1)} P_1$ is a nonzero constant. Therefore by applying $\delta_{j(\deg P_1)} \cdots \delta_{j(1)}$ to both sides of (4), we get

$$T_1(\xi) = C\delta_{j(\deg P_1)} \cdots \delta_{j(1)} (e^{i\alpha\xi} P_2(\xi))\, T_2(\xi) = e^{i\alpha\xi} \tilde{P}_2(\xi)\, T_2(\xi)$$

or

$$e^{-i\alpha\xi} T_1(\xi) = \tilde{P}_2(\xi)\, T_2(\xi).$$

From elementary calculus, we know that $\deg \tilde{P}_2 = 0$ and then Lemma 3 follows. ▮

LEMMA 4. *Let $T$ be a nonzero generalized trigonometric polynomial and $H$ be a nonzero trigonometric polynomial. If*

$$T(\xi) = H(\xi/m)\, T(\xi/m), \tag{5}$$

*then $e^{-i\xi l/m} T(\xi)$ is a trigonometric polynomial for some $l \in Z^n$.*

*Proof.* Write

$$T(\xi) = \sum_j e^{ix_j\xi} T_j(\xi) = \sum_k e^{iy_k\xi} Q_k(\xi), \tag{6}$$

where $T_j(\xi)$ are trigonometric polynomials and $x_j - x_{j'} \notin Z^n$ when $j \neq j'$, and $Q_k(m\xi)$ are trigonometric polynomials and $y_k - y_{k'} \notin Z^n/m$ when $k \neq k'$. Therefore we may write (5) as

$$\sum_k e^{iy_k\xi} Q_k(\xi) = \sum_j e^{ix_j\xi/m} H(\xi/m)\, T_j(\xi/m). \tag{7}$$

For any fixed $k$, we assume that $y_k - x_j/m \in Z^n/m$ for some $j$. Observe that each term in $e^{i\xi x_{j'}/m} H(\xi/m)\, T_{j'}(\xi/m)$ is not a term in $e^{iy_k\xi} Q_k(\xi)$ when $j' \neq j$, and each term in $e^{iy_{k'}\xi} Q_{k'}(\xi)$ is not a term in $e^{ix_j\xi/m} H(\xi/m)\, T_j(\xi/m)$ when $k' \neq k$. It follows from $H \not\equiv 0$ and (7) that

$$e^{iy_k\xi} Q_k(\xi) = e^{ix_j\xi/m} H(\xi/m)\, T_j(\xi/m), \tag{8}$$

and $\#\{y_k\} = \#\{x_j\}$. Therefore by (6)

$$T(\xi) = \sum_j e^{i\xi x_j} T_j(\xi) \tag{9}$$

with $x_j - x_{j'} \notin Z^n/m$ when $j \neq j'$. By (8), there furthermore exists $x_{j'}$ and $s \in Z^n$ for any $x_j$ in (9) such that $x_j = x_{j'}/m + s/m$ and

$$e^{i\xi x_j} T_j(\xi) = e^{ix_{j'}\xi/m} H(\xi/m)\, T_{j'}(\xi/m). \tag{10}$$

Define a map $M$ on $\{x_j\}$ by

$$M(x_j) = x_{j'},$$

where $x_{j'}$ is chosen as above. Then $M$ is well-defined and $M$ is one-to-one on $\{x_j\}$. Define $X_s = \{M^k x_s;\ k = 1, 2, ...\}$ for every $x_s$. Then $X_s = X_{s'}$ or $X_s \cap X_{s'} = \emptyset$. Then we can choose finite numbers of $X_l$ such that

$$\{x_j\} = \bigcup_l X_l \qquad \text{and} \qquad X_l \cap X_{l'} = \emptyset.$$

Therefore the lemma follows if it is proved that $X_l$ is a singleton for every $l$ and that there is only one $X_l$ in the above decomposition of $\{x_j\}$.

We first prove that for every $l$, $X_l$ has only one element by contradiction. Suppose to the contrary that $X_1 = \{x_1, ..., x_k\}$ for some $k \geqslant 2$ for simplicity. Then we have

$$T_s(\xi) = e^{i\alpha_s \xi} H(\xi/m)\ T_{s+1}(\xi/m) \tag{11}$$

for all $1 \leqslant s \leqslant k$ by (9), where $\alpha_s \in Z^n/m$ and we define $T_1(\xi) = T_{k+1}(\xi)$. Hence we have

$$T_s(\xi) = e^{i\alpha'_s \xi} \prod_{j=1}^{k} H\left(\frac{\xi}{m^j}\right) T_s\left(\frac{\xi}{m^k}\right)$$

for some $\alpha'_s \in Z^n/m^k$. Write $T_s(\xi) = P_s(\xi) + R_s(\xi)$, where each $P_s$ is homogeneous polynomial with degree $K$, $|R_s(\xi)| \leqslant C\,|\xi|^{K+1}$ for bounded $\xi$ and all $1 \leqslant s \leqslant k$, and $P_s$ is nonzero at least for one $1 \leqslant s \leqslant k$. Therefore $H(0)^k = m^{kK}$ and the explicit formula

$$T_s(\xi) = e^{i\alpha'_s (m^k/(m^k-1))\,\xi} g(\xi)\ P_s(\xi) \tag{12}$$

holds for all $1 \leqslant s \leqslant k$, where $g(\xi) = \prod_{j=1}^{\infty} \{H(\xi/m^j)/H(0)\}$. Hence

$$e^{i\alpha'_s (m^k/(m^k-1))\,\xi} P_s(\xi)\ T_1(\xi) = e^{i\alpha'_1 (m^k/(m^k-1))\,\xi} P_1(\xi)\ T_s(\xi)$$

for all $2 \leqslant s \leqslant k$. Furthermore there exist $j_s \in Z^n$ and nonzero $c_s$ such that

$$P_s(\xi) = c_s P_1(\xi)$$

and

$$T_s(\xi) = c_s e^{ij_s \xi} T_1(\xi) \tag{13}$$

for all $1 \leqslant s \leqslant k$ by Lemma 3. After choosing $x_j$ appropriately in (9), we may assume $j_s = 0$ in (13). Therefore we have

$$\begin{aligned} c_s e^{ix_s \xi} T_1(\xi) &= e^{ix_s \xi} T_s(\xi) \\ &= e^{ix_{s+1} \xi/m} H(\xi/m)\ T_{s+1}(\xi/m) \\ &= e^{ix_{s+1} \xi/m} H(\xi/m)\ T_1(\xi/m)\ c_{s+1} \end{aligned}$$

by (8) and (13), and $x_s-(x_{s+1}/m)=j/m$ for some fixed $j\in Z^n$ and all $1\leqslant s\leqslant k$. Recall that $T_1(\xi)=T_{k+1}(\xi)$ and $x_1=x_{k+1}$. Therefore $x_s=(j/(m-1))$ for all $1\leqslant s\leqslant k$, which contradicts the fact that $x_j-x_{j'}\notin Z^n/m$ when $j\neq j'$ in (9). This prove that $X_l$ has only one element for every $l$.

We next prove that there is only one $X_l$ in the decomposition of $\{x_j\}$ by contradiction. Assume that the only element in $X_l$ is just $x_l$ without loss of generality since $X_l$ has only one element for every $l$. Hence

$$e^{ix_j\xi}T_j(\xi)=e^{ix_j\xi/m}H(\xi/m)\,T_j(\xi/m)$$

by (10), and

$$T_j(\xi)=e^{i\alpha_j^*\xi}g(\xi)\,P_j(\xi) \tag{14}$$

by (12) for some $\alpha_j^*\in R^n$. Therefore we get $T_j(\xi)=c_je^{ik_j\xi}T_1(\xi)$ for some $k_j\in Z^n$ and nonzero constants $c_j$ by Lemma 3. After choosing $x_j$ appropriately, we may assume $k_j=0$. Then

$$e^{ix_j\xi}T_1(\xi)=e^{ix_j\xi/m}H(\xi/m)\,T_1(\xi/m)$$

for all $j$, and $x_j-x_1\in Z^n$, which contradicts (9), since $x_j-x_1\notin Z^n/m$. ∎

Now we start to prove Theorems 1 and 2.

*Proof of Theorem* 1. Necessity. Let $P$ be a homogeneous polynomial of degree $K$. Define $\tilde{H}(z)=m^{K+N}R(z^m)/R(z)\prod_{j=1}^N(z^{ma_j}-1)/(z^{a_j}-1)$. Then we have

$$\hat{f}(\xi)=\tilde{H}(e^{i\xi/m})\,\hat{f}(\xi/m)$$

or

$$f(x)=\sum_{j\in Z^n}c_jf(mx-j),$$

where $\sum_{j\in Z^n}c_jz^j=\tilde{H}(z)$. The necessity is proved.

Sufficiency. Let $f$ be a blockwise polynomial that satisfies the refinement equation (1). Define

$$H(\xi)=m^{-n}\sum_{j\in Z^n}c_je^{-ij\xi}.$$

Then

$$\hat{f}(\xi)=H(\xi/m)\,\hat{f}(\xi/m) \tag{15}$$

by taking Fourier transform on both sides of (1). By Lemma 4,

$$\hat{f}(\xi)=\sum_j \frac{T_j(\xi)}{P_j(\xi)}=\sum_{s\geqslant s_0}\sum_{\deg P_j=s}\frac{T_j(\xi)}{P_j(\xi)} \tag{16}$$

for some integer $s_0\geqslant 0$, where $\sum_{\deg P_j=s_0}(T_j(\xi)/P_j(\xi))\not\equiv 0$ and $\{P_j(\xi)^{-1}\}_{\deg P_j=s}$ is *linearly independent* for every nonnegative integer $s$, i.e., $\sum_{\deg P_j=s} d_j P_j(\xi)^{-1}=0$ holds only when $d_j=0$. Observe that

$$\sum_{s>s_0}\sum_{\deg P_j=s}\frac{T_j(r\xi)}{P_j(r\xi)}r^{s_0}\to 0 \qquad \text{as} \quad r\to+\infty \qquad \text{a.e.} \quad \xi\in S^{n-1}.$$

Here $S^{n-1}=\{x\in R^n,\ |x|=1\}$ is the unit sphere in $R^n$ and a.e. denotes *almost everywhere*. Therefore we get

$$\sum_{\deg P_j=s_0}\frac{T_j(r\xi)-m^{s_0}H(r\xi/m)\,T_j(r\xi/m)}{P_j(\xi)}\to 0 \qquad \text{as} \quad r\to+\infty \qquad \text{a.e.}$$

for $\xi\in S^{n-1}$. Write

$$T_j(\xi)-H(\xi/m)\,m^{s_0}T_j(\xi/m)=\sum_k c_{jk}e^{iy_k\xi}$$

and let

$$D_k(\xi)=\sum_{\deg P_j=s_0}\frac{c_{jk}}{P_j(\xi)}.$$

Observe that $y_k\xi\neq y_{k'}\xi$ a.e. for $\xi\in S^{n-1}$ when $k\neq k'$. Hence we get $D_k(\xi)=0$ a.e. for $\xi\in S^{n-1}$ by Lemma 2 since $\sum_k D_k(\xi)\,e^{iy_k\xi r}\to 0$ as $r\to+\infty$ a.e. for $\xi\in S^{n-1}$. Recall that $\{P_j(\xi)^{-1}\}$ is linearly independent and $P_j$ are homogeneous polynomials of degree $s_0$. Therefore $c_{jk}=0$ and $T_j(\xi)=m^{s_0}H(\xi/m)\,T_j(\xi/m)$ for all $j$ with $\deg P_j=s_0$. Inductively we can prove

$$T_j(\xi)=m^{\deg P_j}H(\xi/m)\,T_j(\xi/m) \tag{17}$$

for all $j$ and

$$T_j(\xi)=e^{i\alpha_j\xi}g(\xi)\,Q_j(\xi)$$

as in the proof of Lemma 4 (see (14)), where $\deg Q_j-\deg P_j$ is a fixed integer. Recall that $T_j\not\equiv 0$ for all $\deg P_j=s_0$. Therefore we get $T_j(\xi)=c_je^{i(l/(m-1))\,\xi}\tilde{T}(\xi)$ for all $j$ with $\deg P_j=s_0$ by Lemma 4 and we get $T_j(\xi)=0$

for all $j$ with $\deg P_j > s_0$ by Lemma 3, since $\deg Q_j \neq \deg Q_{j'}$ when $\deg P_j \neq \deg P_{j'}$. Furthermore

$$\hat{f}(\xi) = \sum_{\deg P_j = s_0} c_j/P_j(\xi)\, e^{i(l/(m-1))\,\xi}\tilde{T}(\xi).$$

Write

$$\sum_{\deg P_j = s_0} c_j/P_j(\xi) = P(\xi)/Q(\xi) \tag{18}$$

such that $Q$ and $P$ has no common factors, where $Q$ is a principal homogeneous polynomial and $P$ is a homogenous polynomial. Then we get

$$Q(\xi)\, \hat{f}(\xi) = e^{i(l/(m-1))\,\xi}\tilde{T}(\xi)\, P(\xi) \tag{19}$$

for all $\xi \in R^n$. Let $Q(\xi) = \prod_{j=1}^N a_j \xi$ with $0 \neq a_j \in R^n$. Then

$$\tilde{T}(\xi) = 0 \tag{20}$$

on the hyperplanes $a_j \xi = 0$ for all $1 \leqslant j \leqslant N$ from (19) and the continuity of $\hat{f}$. Now we prove that for any fixed $1 \leqslant j \leqslant N$ there exists constant $\alpha_j \in R$ such that $\alpha_j a_j \in Z^n$ and

$$\tilde{T}(\xi) = (e^{i\alpha_j a_j \xi} - 1)\, \tilde{T}_j(\xi). \tag{21}$$

Let $A_j$ be a matrix such that $\det A_j = 1$ and $a_j = (0, ..., 0, 1)\, A_j^{-1}$. Write $\tilde{T}(\xi) = \sum_s t_s e^{is\xi}$. Then (20) implies that $\sum_s t_s e^{isA_j(\xi', 0)} = 0$, where $\xi' = (\xi_1, ..., \xi_{n-1}) \in R^{n-1}$. For typographical reasons, we also use $(\xi_1, ..., \xi_n)$ to stand for the transpose of $(\xi_1, ..., \xi_n)$ when there is no chance of confusion. Write $sA_j(\xi', 0) = x_s \xi'$. Observe that $\sum_s t_s e^{ix_s \xi'} = 0$ implies $t_s = 0$ if $x_s \neq x_{s'}$ for all $s \neq s'$, which contradicts $\tilde{T}(\xi) \not\equiv 0$. Hence there exist numbers $s \neq s' \in Z^n$ such that $(x_s - x_{s'})\, \xi' = (s - s')\, A_j(\xi', 0) = 0$ for all $\xi' \in R^{n-1}$ and $(s - s')\, A_j = (\beta_j)^{-1}\, (0, ..., 0, 1) \neq 0$ for some $\beta_j$. Therefore $a_j = \beta_j(s - s') \neq 0$ for some $\beta_j \in R$. Let $\alpha_j \in R$ be the real number such that $\alpha_j a_j \in Z^n$ and $\alpha_j a_j \notin kZ^n$ for all integers $k$ with $|k| > 1$. Let $B_j$ be a matrix with integral entries whose determinant is 1 and its last column is $\alpha_j a_j$. Let $\tilde{T}(B_j^{-1}\eta) = \sum_{k \in Z} e^{ik\eta_n} Q_k(\eta')$ where $\eta = B_j \xi$. Then $\sum_k Q_k(\eta') = 0$ for all $\eta' \in R^{n-1}$ by (20) and

$$\tilde{T}(B_j^{-1}\eta) = \sum (e^{ik\eta_n} - 1)\, Q_k(\eta') = (e^{i\eta_n} - 1)\, \bar{T}(\eta', \eta_n).$$

Equation (21) is proved. By induction we can prove that

$$\tilde{T}(\xi) = \prod_{j=1}^N (e^{i\alpha_j a_j \xi} - 1)\, R(\xi) \tag{22}$$

after a finite number of steps, where $R(\xi)$ is a trigonometric polynomial. This proves that there exist $\tilde{a}_j \in Z^n$, $l \in Z^n$, homogeneous polynomial $P$ and trigonometric polynomial $R$ such that

$$\hat{f}(\xi) = \prod_{j=1}^{N} \left( \frac{e^{i\tilde{a}_j \xi} - 1}{i\tilde{a}_j \xi} \right) R(\xi)\, P(\xi)\, e^{i(l/(m-1))\,\xi}$$

and

$$f(x) = P(D) \left( \sum_{k \in Z^n} d_k B_{\Xi} \left( x - k - \frac{l}{m-1} \right) \right)$$

by combining (19) and (22), where $\Sigma_{k \in Z^n} d_k e^{-ij\xi} = R(\xi)$. Let $\tilde{R}(z) = \sum_{k \in Z^n} d_k z^k$. Then

$$\tilde{R}(z^m) \prod_{i=1}^{N} (z^{m\tilde{a}_j} - 1) = m^{N + \deg P} \tilde{H}(z)\, \tilde{R}(z) \prod_{i=1}^{N} (z^{\tilde{a}_j} - 1)$$

by (14), where $\tilde{H}(z) = m^{-n} \sum_{j \in Z^n} c_j z^j$. Observe that $f$ is supported on a hyperplane when $\operatorname{rank}(\tilde{a}_1, ..., \tilde{a}_N) \leqslant n-1$. Then $\operatorname{rank}(\tilde{a}_1, ..., \tilde{a}_N) = n$ when $f$ is a nonzero function. The sufficiency and hence Theorem 2 is proved. ∎

*Proof of Theorem* 2. The necessity is proved in [LLS].

Sufficiency. Let $f$ be smooth on $(a_j, a_{j+1})$ $(1 \leqslant j \leqslant N)$ and $\operatorname{supp} f \subset [a_1, a_{N+1}]$. Define $(d/dx)^k f_-(a_j) = \lim_{x \to a_j,\, x < a_j} (d/dx)^k f(x)$, $(d/dx)^k f_+(a_j) = \lim_{x \to a_j,\, x > a_j} (d/dx)^k f(x)$ and $f_k(a_j) = (d/dx)^k f_+(a_j) - (d/dx)^k f_-(a_j)$. By integration by parts we get

$$\hat{f}(\xi) = \sum_{k=0}^{M} (i\xi)^{-k} \sum_{j=1}^{N+1} f_k(a_j)\, e^{-ia_j \xi} + (i\xi)^{-M} \sum_{j=1}^{N} \int_{a_j}^{a_{j+1}} e^{-ix\xi} \left( \frac{d}{dx} \right)^{M+1} f(x)\, dx$$

for every integer $M \geqslant 1$. Let $T_k(\xi) = \sum_{j=1}^{N+1} f_k(a_j)\, e^{-ia_j \xi}$. By the same procedure used in the proof of Theorem 1, we can prove $T_k(\xi) = 0$ except when $k = k_0$ for some nonnegative integer $k_0$ and

$$T_{k_0}(\xi) = m^{k_0} H(\xi/m)\, T_{k_0}(\xi/m). \tag{23}$$

Therefore $f_k(a_j) = 0$ or $\lim_{x \to a_j} (d/dx)^k f(x)$ exists for all $a_j$ when $k > k_0$ since $T_k(\xi) = 0$. Define $h(x) = (d/dx)^{k_0 + 1} f(x)$ when $x \neq a_j$ for all $j$ and $h(x) = \lim_{x \to a_j} (d/dx)^{k_0 + 1} f(x)$ when $x = a_j$ for some $j$. Then we have

$$\hat{f}(\xi) = (i\xi)^{-k_0}\, T_{k_0}(\xi) + (i\xi)^{-k_0}\, \hat{h}(\xi).$$

Observe that $h \in C^\infty$ has compact support and $h$ satisfies the refinement equation $h(x) = m^{k_0} \sum_j c_j h(mx - j)$ by (23) and (15). Therefore by the non-existence of a $C^\infty$ refinable function with compact support proved by Daubechies and Lagarais [DL] (or Theorem 3), we get $h(x) = 0$. This shows that

$$\hat{f}(\xi) = (i\xi)^{-k_0} T_{k_0}(\xi),$$

and Theorem 2 follows by using the same method as in the proof of Theorem 1. ▌

## 3. PROOF OF THEOREM 3

To prove Theorem 3, we need a lemma.

Define the Zak transform by

$$Z(f)(x, \xi) = \sum_k f(x+k)\, e^{-ik\xi} \tag{24}$$

and define the symbol function of the refinement equation (1) by

$$H(\xi) = \frac{1}{m^n} \sum_{j \in Z^n} c_j e^{-ij\xi}. \tag{25}$$

LEMMA 5. *Let $f$ satisfy* (1). *Then the formula*

$$\sum_{e_l} H((\xi + 2e_l\pi)/m)\, e^{ie_{l'}(\xi + 2e_l\pi)/m} Z(f)(x, (\xi + 2e_l\pi)/m) = Z(f)((x + e_{l'})/m, \xi) \tag{26}$$

*holds for each $e_{l'}$, where $\{e_l\}$ is the set*

$$\{(x_1, x_2, ..., x_n) \in Z^n : 0 \leqslant x_j \leqslant m-1,\ 1 \leqslant j \leqslant n\}.$$

*Proof of Lemma* 5. Recall that

$$\sum_{e_l} e^{i2ke_l\pi/m} = \begin{cases} 0 & k \notin mZ^n \\ m^n & k \in mZ^n \end{cases}$$

for every $k \in Z^n$ and

$$H(\xi) = m^{-n} \sum_j c_j e^{-ij\xi}.$$

Therefore the left-hand side of (26) equals

$$m^{-n} \sum_k \sum_j c_j f(x+k)\, e^{-i(j+k-e_{l'})\,\xi/m} \sum_{e_l} e^{-i(j+k-e_{l'})\, 2e_l \pi/m}$$

$$= \sum_r \sum_j c_j f(x+mr+e_{l'}-j)\, e^{-ir\xi}$$

$$= \sum_r f((x+e_{l'})/m+r)\, e^{-ir\xi}$$

$$= Z(f)((x+e_{l'})/m, \xi)$$

and Lemma 5 is proved. ∎

*Proof of Theorem* 3. Define a linear operator $I$ on $2\pi Z^n$ periodic function by

$$I\colon F(\xi) \to \sum_l H((\xi+2e_l\pi)/m)\, F((\xi+2e_l\pi)/m).$$

Observe that $\int_{[0,\,2\pi]^n} IF(\xi)\, d\xi = (2\pi)^n \sum_k c_k \hat{F}(k)$ where $\hat{F}(k) = (1/(2\pi)^n) \int_{[0,\,2\pi]^n} e^{-ik\xi} F(\xi)\, d\xi$ is the $k$th Fourier coefficient of $F$. Therefore

$$\{IF=0\} \subset \left\{F;\ \sum_k c_k \hat{F}(k) = 0\right\}. \tag{27}$$

Let the set of homogeneous polynomials $\{P_j\}$ be a basis of $\mathscr{P}$. Define $Z^*(f)(\xi) = Z(f)(0, \xi)$. Then Theorem 3 follows easily from (27) and

$$\sum_j c_j Z(P_j(D)\, f) \in \{F;\ IF=0\} \qquad \text{hold only when} \quad c_j = 0 \quad \text{for all } j. \tag{28}$$

Observe that

$$IZ^*(P(D)\, f)(\xi) = m^{\deg P} Z^*(P(D)\, f)(\xi)$$

when $P(D)\, f$ is continuous. Therefore (28) is reduced to

$$Z^*(P^k(D)\, f) = 0 \qquad \text{implies} \quad P^k = 0 \tag{29}$$

for all nonnegative integers $k$, where $P^k = \sum_{\deg P_j = k} c_j P_j$. By the definition of $Z^*(P^k(D)\, f)$, we know that $P^k(D)\, f(j) = 0$ and furthermore that $P^k(D)\, f(x) = 0$ for all $x \in R^n$ by Lemma 5 and from the continuity of $P(D)\, f$. On the other hand, continuity of $\hat{f}$ and $P(i\xi)\, \hat{f}(\xi) = 0$ implies $f = 0$, which contradicts our assumption. Thus (29) is proved, and hence also Theorem 3. ∎

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